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(6) ON THE STRUCTURE OF CONTINUA OF POSITIVE AND CONCAVE SOLUTIONS FOR TWOPOINT NONLINEAR EIGENVALUE PROBLEMS.

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ABSTRACT

We study the eigenvalue problem

$$(P)\begin{cases} -u'' = \lambda g(x,u,u'), & x \in \,]a,b[, \quad u \in C^2[a,b], \ \lambda > \lambda_0 \\ u(a) - cu'(a) = 0, & c,d \geq 0 \\ u(b) + du'(b) = 0\ . \end{cases}$$

We investigate the continuation and the asymptotic behaviour as $\lambda \to \infty$ of positive and concave solutions of (P). In the autonomous case we show that every positive and concave solution (λ,u) can be continued. Concerning the asymptotic behaviour we show that the positive and concave solutions converge to a solution $\bar{u} \neq 0$ of $g(x,u,u') = 0$, if they are uniformly bounded in $C[a,b]$. In the autonomous case the limit $\bar{u}$ is unique. The boundary conditions satisfied by $\bar{u}$ are also discussed.

AMS(MOS) Subject Classification - 34B15, 34E15

Key Words - Two point nonlinear eigenvalue problems, concavity, singular perturbations, continuation.

Work Unit Number 1 - Applied Analysis

EXPLANATION

Physical problems often lead to second order nonlinear boundary value problems where a parameter λ is involved (for example λ is the temperature or the energy). In this paper, we prove that under suitable conditions on the nonlinearity there must be positive and concave solutions for every λ bigger than some λ_0. Moreover these solutions converge to a unique limiting function.

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ON THE STRUCTURE OF CONTINUA OF POSITIVE AND CONCAVE SOLUTIONS FOR TWOPOINT NONLINEAR EIGENVALUE PROBLEMS

Ph. Clément and I. B. Emmerth

I. Introduction

In this paper we consider the following two point eigenvalue problem:

$$(P)\quad \begin{cases} -u'' = \lambda g(x,u,u'), & x \in\,]a,b[, \quad u \in C^2[a,b] \\ u(a) - cu'(a) = 0, & c,d \geq 0, \\ u(b) + du'(b) = 0, & \end{cases}$$

where g is given. We shall consider two cases (forced and bifurcation case), i.e.

A. (H1) $g \in C([a,b] \times \mathbb{R}^2)$, (H2) $g(x,0,0) > 0$, $x \in [a,b]$

B. (H1) $g(x,u,v) = f(x,u,v)u$, $f \in C([a,b] \times \mathbb{R}^2)$,

(H2) $f(x,0,0) > 0$, $x \in [a,b]$.

In both cases under the assumptions (H1) and (H2) it is known [5,10], that there exist two maximal continua of positive (resp. negative) solutions of (P) in $\mathbb{R} \times C^2[a,b]$, unbounded in $\mathbb{R} \times C^1[a,b]$ such that their intersection is $(0,0)$ or $(\lambda_1,0)$, where λ_1 in the bifurcation case is the first positive eigenvalue of the linearized problem at the origin. In what follows, we shall restrict ourselves to the continuum of positive solutions, denoted by $\mathcal{C}$, since the same arguments can be applied to the negati[illegible]tions. Observe that if $(\lambda,u) \in \mathcal{C}$, then $\lambda \geq 0$.

The aim of this paper is to study the asymptotic behaviour of $\mathcal{C}$ as λ tends to ∞ and the continuation of solutions in $\mathcal{C}$. We do this under the assumption, which we call condition (*), that every solution of $\mathcal{C}$ is not only positive, but concave. It must be noted that condition (*) is satisfied in the interesting case, where $g(x,u,v) = s(x)\, h_1(v)\, h_2(u)$, $s > 0$. (See example V.6). However in the general case it can be violated. (See Section IV, counterexample).

Our main results are stated in Theorems IV.3 and IV.5, where sufficient conditions on g are given to insure condition (*), in Theorem II.1, IV.6, IV.7, and IV.8 where the

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asymptotic behaviour of $\mathcal{C}$ is studied and in Theorem III.2, where the continuation is investigated. We apply those theorems to examples in Section V.

Concerning the asymptotic behaviour of $\mathcal{C}$ several possibilities occur. If λ is bounded, there is bifurcation from infinity, see [11], which we do not discuss. Otherwise the interesting case is where u is bounded and λ tends to infinity. This leads to the singular perturbation problem

$$\begin{cases} -\varepsilon u'' = g(x,u,u') \\ u(a) - cu'(a) = 0, \quad c,d \geq 0 \\ u(b) + du'(b) = 0 \ . \end{cases}$$

In Theorem II.1 we only assume a bound on u but not on u' and we get the existence of a positive concave solution $\bar{u} \neq 0$ of the reduced equation $g(x,u,u') = 0$ which is an asymptotic solution to the solutions in $\mathcal{C}$. $\bar{u}$ does not necessarily satisfy the boundary conditions. It must be noted that the derivative of $\bar{u}$ is not continuous on $]a,b[$ in most cases. It will be clear from the examples given in Section V and Theorem I how our results can be applied.

A similar approach for the study of the asymptotic behaviour of $\mathcal{C}$ as $\lambda \to \infty$ is done in [7], where a specific differential equation involving a delay is treated. However in [7] the existence of solutions for small ε is proved by the method of monotone iterations and the convergence to a limit is proved by Laplace's method. With a slight modification that problem can be treated also in our framework. We did not find a similar approach in the literature.

In Theorem III.2 we prove that if in (P) g does not depend on x and is continuously differentiable (autonomous case), then each positive concave solution can be continued locally. In particular if condition (*) is satisfied, $\mathcal{C}$ is a curve, hence there is no secondary bifurcation. In [3] a result of the same nature is proved for the equation $f(u,u',u'') = \lambda u$ with Dirichlet boundary conditions only.

(In [3], the following corrections have to be made. Read $\mathbb{R} \times C^2[a,b]$ instead of $\mathbb{R} \times C^{2+k}[a,b]$ in the statements of Theorems I and I'. Set $k = 0$ in the proofs.)

On the other hand we use Theorem III.2 in Corollary IV.4 to establish condition (*),

and in Theorem IV.8 to prove the uniqueness of ε asymptotic solutions, in the autonomous case.

Several authors [1,8,12] have used continuation arguments for this problem, but our case is not covered by their results.

Theorem I. Let the problem

$$\text{(I.1)}\quad \begin{cases} -u'' = \lambda f(u,u')u , \\ u(a) - cu'(a) = 0, \quad c,d \geq 0 \\ u(b) + du'(b) = 0 , \end{cases}$$

with a) $f \in C^1(\mathbb{R}^2)$,

b) $f(0,0) > 0$,

c) $f_u(u,u') \leq 0$,

d) $\exists\ v_2 < 0 < v_1$ such that $f(0,v_2) = f(0,v_1) = 0$, where v_1 (resp. v_2) is the first positive (resp. negative) zero of $f(0,v)$.

Then 1° $\mathcal{C}$ is a C^1-manifold in $\mathbb{R} \times C^2[a,b]$ and $\mathrm{Proj}_{\mathbb{R}}\, \mathcal{C} \supset [\lambda_1, \infty[$,

2° there exists $\bar{u} \in C[a,b]$, $\bar{u} > 0$ on $]a,b[$ and concave such that

$$f(\bar{u}(x), \bar{u}'_{\pm}(x)) = 0, \quad x \in [a,b] \qquad \bar{u}(a) = \alpha_0, \quad \bar{u}(b) = \alpha_1$$

and $\displaystyle\lim_{\substack{\lambda\to\infty \\ (\lambda,u)\in\mathcal{C}}} \max_{x\in[a,b]} |u(x) - \bar{u}(x)| = 0$,

$\displaystyle\lim_{\substack{\lambda\to\infty \\ (\lambda,u)\in\mathcal{C}}} |u'(x) - \bar{u}'(x)| = 0$ for every point of continuity of $\bar{u}'(x)$ where $\alpha_0 = 0$ if $c = 0$ and

$$\alpha_0 = \sup\{\alpha \mid f(\alpha, tc^{-1}\alpha) > 0,\ t \in [0,1]\} \text{ if } c > 0 ,$$

$$\alpha_1 = 0 \text{ if } d = 0 \text{ and}$$

$$\alpha_1 = \sup\{\alpha \mid f(\alpha, t\, d^{-1}\alpha) > 0,\ t \in [-1,0]\} \text{ if } d > 0 .$$

3° Moreover if $c > 0$ (resp. $d > 0$) and α_0 (resp. α_1) is the first positive

zero of $f(\alpha, c^{-1}\alpha)$ (resp. of $f(\alpha, d^{-1}\alpha)$) then $\bar{u}(a) - c\bar{u}'(a) = 0$ (resp. $\bar{u}(b) - d\bar{u}'(b)) = 0$.

The proof is a straightforward application of our theorems. From (b) and (c) one easily checks that the condition (c2) following Theorem IV.3 is satisfied, thus Corollary IV.4 can be applied. Theorem III.2 gives the first assertion of 1° . Corollary IV.4 again together with (d) provide a bound in C^1 for the solutions belonging to $\mathcal{C}$, therefore from the alternative of Rabinowitz's Theorem $\mathrm{Proj}_{\mathbb{R}}\mathcal{C} \supseteq [\lambda, \infty[$ which completes (1). Now Theorem II.1, IV.7 and IV.8 give 2° . Theorem IV.6 proves 3°.

Remark I.1. It follows from our theory that if $\tilde{f}$ satisfies (a), (b), (d) and for every $v_2 < v < v_1$ $\tilde{f}(\cdot, v)$ and $f(\cdot, v)$ have the same first positive zeros, then all the conclusions of Theorem I hold with the same $\bar{u}$.

Remark I.2. It also follows from our theory, that if in problem (I.1) we replace $f(u,u')u$ by $\tilde{f}(u,u')$ as above (forced case) the same conclusions hold again with the only difference that $\mathrm{Proj}_{\mathbb{R}}\mathcal{C} = [0, \infty[$.

Notations: Throughout this paper we shall use the following notations: $]a,b[$ is an open interval. For $u \in C^k[\alpha,\beta]$ we use $u_{k_1[\alpha,\beta]} := \max_{x \in [\alpha,\beta]} |u^{(k)}(x)|$. If $[\alpha,\beta] = [a,b]$ we shall drop the index $[\alpha,\beta]$. $W^{m,p}[\alpha,\beta]$ are the usual Sobolev spaces. $AC[\alpha,\beta] = W^{1,1}[\alpha,\beta]$, $H^1[\alpha,\beta] = W^{1,2}[\alpha,\beta]$. u is called positive on $]a,b[$ if $u(x) > 0$, $x \in]a,b[$. u is called decreasing if it is nonincreasing. $u'_{\pm}(x)$ denotes the right (resp. left) derivative of x if it exists. $u(x^{\pm})$ denotes the right (resp. left) limit of u at x if it exists. By a curve in $\mathbb{R} \times C^2[a,b]$ we mean a C^1 one-dimensional manifold in $\mathbb{R} \times C^2[a,b]$. $\mathrm{Proj}_{\mathbb{R}}\mathcal{C} := \{\lambda \in \mathbb{R} \mid (\lambda,u) \in \mathcal{C}\}$.

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II. Asymptotic behaviour

In this section we shall use the following notations:

$I := [a,b], \ ]a,b], \ [a,b[, \ ]a,b[$

$E(I) := \{u \in C(I) \mid u \geq 0, \ u \text{ concave}\}$

Let $(u_n)_{n \in \mathbb{N}} \subset E(I) \cap C^1(I)$ and $\bar{u} \in E(I)$. We shall say that u_n converges to $\bar{u}$ in $E(I)$ if

a) $\lim_{n \to \infty} |u_n - \bar{u}|_{AC[K]} = 0$ for every compact interval $K \subset I$

b) there exists $\bar{v}$ decreasing on I such that $u_n'(x) \to \bar{v}(x), \ x \in I$.

Theorem II.1. Let (H1) and (H2) be satisfied. Let $(\lambda_n, u_n)_{n \in \mathbb{N}}$ be a sequence of positive and concave solutions of (P). Let $M > 0$ such that

a) $\lambda_n \to \infty$ as $n \to \infty$,

b) $|u_n|_0 \leq M$ for every n.

Then there exists $\bar{u}$ positive and concave on $]a,b[$ satisfying

1) $g(x, \bar{u}(x), \bar{u}_+'(x)) = 0, \ x \in]a,b[$,

2) $g(x, \bar{u}(x), \bar{u}_-'(x)) = 0$

and there exists a subsequence $(\lambda_{n_k}, u_{n_k})_{k \in \mathbb{N}}$ such that $(u_{n_k})_{k \in \mathbb{N}}$ converges to $\bar{u}$ in $E(]a,b[)$.

If moreover

c) $u_n'(a) \leq M$

then $\bar{u}$ belongs to $E([a,b[)$, $g(a, \bar{u}(a), \bar{u}_+'(a)) = 0$ and $(u_{n_k})_{k \in \mathbb{N}}$ converges to $\bar{u}$ in $E([a,b[)$.

Remarks II.1.

a) In the bifurcation case $(g(x,u,u') = f(x,u,u')u$ obviously $f(x,\bar{u}(x), \bar{u}_+'(x)) = 0$, $x \in]a,b[$.

b) In the case where condition c) holds, since $\bar{u} \in C([a,b[)$, $\bar{u}$ satisfies the boundary condition $\bar{u}(a) = 0$ when $c = 0$. However if $c > 0$, the boundary condition is not necessarily preserved. For a detailed discussion see Theorems IV.6 and IV.7.

c) If condition (c) is replaced by $u'(b) \geq -M$, $E([a,b[)$ has to be replaced by $E(]a,b])$ in the conclusions.

d) In the case, where $|u|_1 \leq M$ (respectively $|u|_0 \leq M$ when g does not depend on u'). Condition (a) is automatically satisfied.

e) When g depends on u' it can happen that $|u_n|_1$ is unbounded and nevertheless conditions (a) and (b) are satisfied. Therefore, even if the bifurcation diagram in $\mathbb{R} \times C^1[a,b]$ would suggest the absence of a limiting function, a limit still exists in the sense of Theorem (II.1). For an example see Section V.

f) In order to apply Theorem II.1 to a concrete case, a bound on the solutions has to be found. This question as well as condition (*) will be discussed in Section IV.

<u>Proof of Theorem II.1.</u> The proof is done in these steps. First we prove that there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ such, that $g(x,u_{n_k},u'_{n_k}) \to 0$ a.e. Then using Theorem A, Appendix we extract a subsequence, still denoted by u_{n_k}, converging to some $\bar{u}$ in an appropriate sense, which will imply that $g(x,\bar{u},\bar{u}') = 0$. Finally we show that $\bar{u} \neq 0$ (which in the forced case is obvious). As it will be clear from the proof, the concavity of u_n plays an essential role in each step.

Let assumptions (a) and (b) be satisfied, i.e. $\lambda_n \to \infty$ and $|u_n|_0 \leq M$, $n = 1,2,\dots$. In this case we denote by I the open interval $]a,b[$. If moreover assumption (c) holds, i.e. $u'_n(a) \leq M$, then $I := [a,b[$. Similarly if $u'_n(b) \geq -M$ then $I :=]a,b]$, if $u'_n(a) \leq M$ and $u'_n(b) \geq -M$ then $I := [a,b]$. This enables us to treat all cases at the same time. By $K := [\alpha,\beta]$ we denote any closed subinterval of I .

We claim that there exists $M(K)$ such that

$$|u_n|_{1,K} \leq M(K). \tag{II.1}$$

Indeed we have $-u''_n(x) \geq 0$ since u_n is concave. u'_n being decreasing $|u_n|_{1,K} = \max\{|u'_n(\alpha)|, |u'_n(\beta)|\}$. Assume $|u'_n(\alpha)|$ is unbounded. By the definition of I it follows that $\alpha \neq a$. If some subsequence $u'_{n_k}(\alpha) \to +\infty$, then we have $u'_{n_k}(x) \to +\infty$ for $a \leq x \leq \alpha$, hence $u_n(\alpha) = \int_a^\alpha u'_{n_k}(x)dx + u_n(a) \to +\infty$ in contradiction to (b) . The other cases are similar.

(i) <u>There exists a subsequence</u> $(u_{n_k})_{k \in \mathbb{N}}$ <u>such that</u> $g(x,u_{n_k},u'_{n_k}) \to 0$ <u>a.e.</u> <u>on</u> $[a,b]$. By (II.1) and the fact that $-u''_n(x) \geq 0$ we have $|u''_n|_{L^1(K)} \leq M(K)$. Hence $|g(\cdot,u_n,u'_n)|_{L^1(K)} \to 0$ since $\lambda_n \to \infty$ and $(\lambda_n,u_n) \in C_1$. Choose $K_1 \subset K_2 \subset \dots \subset I$

such that $\cup K_m = I$. One completes the proof by extracting successive subsequences converging a.e. on K_m .

(ii) <u>There exist</u> $\bar{u} \in E(I)$ <u>and a subsequence, still denoted by</u> u_{n_k} , <u>converging to</u> $\bar{u}$ <u>in</u> $E(I)$. Let K_m be as before. On each K_m we have by (II.1) $|u_n|_{1,K_m} \leq M(K_m)$ and $|u_n|_{0,K_m} \leq M$. From proposition A. Appendix we have that there exist $\bar{u}_m \in E(K_m)$ decreasing and a subsequence u_{n_k} such that

$$|u_{n_k} - \bar{u}_m|_{AC[K_m]} \to 0 \ ,$$

$$u'_{n_k}(x) \to \bar{v}_m(x), \quad x \in K_m \ ,$$

$$\bar{v}_m(x^{\pm}) = \bar{u}'_{m\pm}(x), \quad x \in K_m \ .$$

One completes the proof as in (i) by defining $\bar{u}$ and $\bar{v}$ by $\bar{u}|K_m = \bar{u}_m$, $\bar{v}|K_m = \bar{v}$. Observe that $\bar{u} \in E(I)$.

(iii) $g(x,\bar{u},\bar{u}'_{\pm}) = 0, \ x \in I$ <u>and</u> $\bar{u} > 0$. By (H1), (i) and (ii) $g(x,\bar{u},\bar{u}') = 0$ a.e. on I . From the continuity of g and the Appendix we get $g(x,\bar{u}(x), \bar{u}'_{\pm}(x)) = 0, \ x \in I$. Since $\bar{u}$ is concave on $]a,b[$ it is sufficient to prove that $\bar{u} \neq 0$, and hence $\bar{u} > 0$ on $]a,b[$ in the bifurcation case. (In the forced case $\bar{u} \neq 0$ is obvious since $f(x,0,0) > 0$) . To do that we use the following.

<u>Lemma</u>: Let $\rho, p, q \in C[a,b]$ satisfying

a) $\rho(x) > 0, \ x \in [a,b]$,

b) p, q concave, $\neq 0$, $p(a), p(b), q(a), q(b) \geq 0$.

Then

$$\int_a^b \frac{p(x)\ q(x)}{\rho(t)\ p(t)\ q(t)dt} \geq \psi(x), \quad x \in [a,b] \ ,$$

where

$$\psi(x) = \begin{cases} |\rho|^{-1}_{0,[a,b]} \ (b-a)^{-3} \ (x-a)^2, & x \in [a, \frac{a+b}{2}] \ , \\ |\rho|_{0,[a,b]} \ (b-a)^{-3} \ (x-b)^2, & x \in [\frac{a+b}{2}, b] \ . \end{cases}$$

<u>Proof</u>: We have

$$p(x) \geq \begin{cases} p(\xi)\ \frac{x-a}{\xi-a}, & x \in [a,\xi] \ , \\ p(\xi)\ \frac{x-b}{\xi-b}, & x \in [\xi,b] \end{cases}$$

for any $\xi \in]a,b[$. Similarly for q . Choose $\bar{\xi}$ such that $p(\bar{\xi})\, q(\bar{\xi}) = |pq|_{0,[a,b]}$. Clearly $\bar{\xi} \neq a,b$. Hence

$$\frac{p(x)\ q(x)}{\int_a^b \rho(t)\rho(t)q(t)dt} \geq \begin{cases} |\rho|_{0,[a,b]}^{-1} \dfrac{1}{(b-a)} \dfrac{p(\bar{\xi})\ q(\bar{\xi})}{p(\bar{\xi})\ q(\bar{\xi})} \left(\dfrac{x-a}{\bar{\xi}-a}\right)^2, & x \in [a,\bar{\xi}] \\ |\rho|_{0,[a,b]}^{-1} \dfrac{1}{(b-a)} \dfrac{p(\bar{\xi})\ q(\bar{\xi})}{p(\bar{\xi})\ q(\bar{\xi})} \left(\dfrac{x-b}{\bar{\xi}-b}\right)^2, & x \in [\bar{\xi},b] \end{cases}$$

which completes the proof of the lemma.

Let h_1 be the eigenfunction corresponding to λ_1 of the linearized problem at the origin, which we choose to be positive. From (P) we get

$$\varepsilon_k := \frac{\int_a^b f(x,u_{n_k},u'_{n_k})\, u_{n_k}\, h_1\, dx}{\int_a^b f(x,0,0)\, u_{n_k}\, h_1\, dx} = \lambda_1\, \lambda_{n_k}^{-1} \to 0 \ .$$

By the previous lemma and condition (*) we have

$$0 \leq \int_\alpha^\beta f(x,u_{n_k},u'_{n_k})\, \psi(x)dx \leq \varepsilon_k$$

where $[\alpha,\beta] \subseteq I$. Assume $\bar{u} = 0$. Then we have

$$u_{n_k} \to 0 \quad \text{on} \quad [\alpha,\beta]$$

$$u'_{n_k} \to 0 \quad \text{a.e. on} \quad [\alpha,\beta] \quad \text{(even everywhere)}$$

$$|u_{n_k}|_{1,[\alpha,\beta]} \leq M([\alpha,\beta]) \ .$$

By (H1) and Lebesgue's dominated convergence theorem we get

$$\int_\alpha^\beta f(x,0,0)\ \psi(x)dx = 0$$

which contradicts (H2) since $\psi(x) > 0$ on $]a,b[$.

III. Continuation

Theorem III.2. *Let* $g \in C^1(\mathbb{R}^2)$ *and* $u_0 \in C^2[a,b]$, $u_0 \neq 0$ *and* $\lambda_0 \in \mathbb{R}$ *satisfy the problem*

$$\text{(III.1)} \quad \begin{cases} -u'' = \lambda g(u,u')\ , \\ u(a) - cu'(a) = 0, \quad c,d \geq 0\ , \\ u(b) + du'(b) = 0\ , \end{cases}$$

with $\lambda_0 g(u_0,u_0') \geq 0$. *Then there exists a neighbourhood* $\mathcal{U}$ *of* (λ_0,u_0) *in* $\mathbb{R} \times C^2[a,b]$ *and a* C'*-map* $s \to (\lambda(s), u(s))$, $s \in]-1,1[$ *from* $]-1,1[$ *into* $\mathbb{R} \times C^2[a,b]$ *such that*

(a) $(\lambda(0), u(0)) = (\lambda_0,u_0)$

(b) $(\lambda(s), u(s))$ *satisfies problem* (III.1) *for* $s \in]-1,1[$ *and in* $\mathcal{U}$ *there are no other solutions of* (III.1)

Remarks III.

(a) Obviously u_0 is concave on $[a,b]$ and different from zero, hence positive on $]a,b[$ by the boundary conditions. Moreover observe that $u_0'(a)$ and $u_0'(b)$ are different from zero by using again the concavity of u_0, the boundary conditions and $u_0 \neq 0$.

(b) Theorem III.2 can be applied to the investigation of the structure of $\mathcal{C}$ of problem (P), i.e. if (H1) is replaced by the stronger assumption (H'1): $g \in C^1(\mathbb{R}^2)$ and g does not depend on x and condition (H2) and (*) are satisfied, then $\mathcal{C}$ is a curve in $\mathbb{R} \times C^2[a,b]$.

Proof of Theorem III.2. Let $E_1 := \{u \in C^2[a,b] \mid u(a) - cu'(a) = 0,\ u(b) + du'(b) = 0\}$,

$E_2 := C[a,b]$,

Let $F: \mathbb{R} \times E_1 \to E_2$ be defined by

$$F(\lambda,u)(x) := u''(x) + \lambda g(u,u')(x)\ .$$

Then (III.1) is equivalent to the equation $F(\lambda,u) = 0$ for which we know a solution (λ_0,u_0). Remark that $F \in C^1(\mathbb{R} \times E_1,E_2)$, $d_u F(\lambda_0,u_0)$ is Fredholm of index 0 and $\dim N(d_u F(\lambda_0,u_0)) \leq 1$. If $\dim N(d_u F(\lambda_0,u_0)) = 0$ we are done by the implicit function theorem. In the case $\dim N(d_u F(\lambda_0,u_0)) = 1$, by Theorem 3.2 of [6] it is sufficient to show that

$$\text{(III.2)} \qquad d_\lambda F(\lambda_0,u_0) \notin R(d_u F(\lambda_0,u_0))\ .$$

We claim that (III.2) is equivalent to the condition

(III.3) $$\int_a^b r(x)\, g(u,u')(x)\, \tilde{h}(x)dx \neq 0$$

where

$$r(x) := \exp\{\int_a^x \lambda_0\, g_{u'}(u,u')(t)dt\} > 0$$

and

$$\tilde{h} \in \mathbb{N}(d_u\, F(\lambda_0,u_0)), \quad \tilde{h} \neq 0 .$$

Indeed $\overline{g} \in R(d_u\, F(\lambda_0,u_0))$ is equivalent to

(III.4) $$(rh_0')' + \lambda_0 r\, g_u(u_0,u_0')h_0 = r\overline{g}$$

for some $h_0 \in E_1$. It is well known that (III.4) can be solved iff $\int_a^b r\,\overline{g}\,\tilde{h}\,dx = 0$ since

$$(r\tilde{h}')' + \lambda_0\, g_b\, g_u(u_0,u_0')\tilde{h} = 0$$

$$\tilde{h}(a) > 0, \quad \int_a^b \tilde{h}^2(x)dx = 1, \quad \tilde{h} \in E_1 .$$

The proof will be complete if we show that $\tilde{h}(x) \neq 0$ on $]a,b[$. Let $v = u'$. v satisfies

(III.5) $$v'' + \lambda_0\, g_u(u_0,u_0')v + \lambda_0\, g_{u'}(u_0,u_0')v' = 0$$

since $u \in C^3[a,b]$. Observe that $v(a) \neq 0$ and the zeroes of v are simple, since $u \neq 0$, $g \in C^1$ and from the boundary conditions. Since u is concave and by the previous remark, v has exactly one zero on $]a,b[$, hence by remark III.a exactly one zero in $[a,b]$.

Now observe that we have $\tilde{h}$ and v satisfying

(III.6) $$-(rz')' - \lambda_0\, r\, g_u(u_0,u_0')z = \mu z, \quad \mu = 0$$

with different boundary conditions. We shall prove that $\tilde{h}$ has to be the first eigenfunction of (III.6) with the boundary conditions

(BCh) $$z(a) - c\, z'(a) = 0, \quad z(b) + d\, z'(b) = 0 .$$

Thus it is well known, that $\tilde{h} \neq 0$ on $]a,b[$ [4]. We first consider the case of Dirichlet boundary conditions $(c = d = 0)$. In this case $\tilde{h} \neq 0$ on $]a,b[$ follows from Sturm's comparison theorem [4] by using the fact that v has exactly one zero on $]a,b[$, $v(a) \neq 0$, $v(b) \neq 0$ and $\tilde{h}(a) = \tilde{h}(b) = 0$.

Then we consider the case $c,d > 0$. By remark III.a $u_0(a), u_0(b) > 0$. From (III.1) it follows that v satisfies (III.6) and

$$\text{(BCv)} \qquad \begin{cases} z'(a) + c\,\dfrac{g(u,u')}{u(a)}\,z(a) = 0 \\[2ex] z'(b) - d\,\dfrac{g(u,u')}{u(b)}\,z(b) = 0 \end{cases}$$

Let us call $\mu_1^v < \mu_2^v < \ldots$ (resp. $\mu_1^h < \mu_2^h < \ldots$) the eigenvalues of (III.6) with (BCv) (resp. (BCh)). By the fact that v has exactly one zero in $]a,b[$, v is the eigenfunction corresponding to μ_2^v hence $\mu_2^v = 0$ [4]. By using a variational characterization of the μ_i's.

We shall prove that

(i) $\mu_i^v \leq \mu_i^h \quad i = 1,2,\ldots$,

(ii) $\mu_2^h > 0$.

This will complete the proof of this case since $\tilde{h}$ is an eigenfunction of (III.6) and (BCh) with corresponding eigenvalue 0, which implies that $\tilde{h}$ corresponds to μ_1^h. For $z \in H^1[a,b]$ we define

$$Q_n[z] := \int_a^b r(z'^2 - \lambda_0\, g_u(u_0,u_0')z^2)dt \quad ,$$

$$Q_h[z] := Q_n[z] + \{\frac{r(b)}{d}\, z^2(b) + \frac{r(a)}{c}\, z^2(a)\} \quad ,$$

$$Q_v[z] := Q_n[z] - \{r(b)d\,\frac{g(u_0,u_0')}{u_0(b)}\, z^2(b) + r(a)c\,\frac{g(u_0,u_0')}{u_0(a)}\, z^2(a)\} .$$

Let $\mathcal{E} := \{E \subseteq H^1[a,b] \mid E = F \cap B,\ F \text{ closed supspace of } H^1[a,b],\ \operatorname{codim} F = 1\}$, where $B := \{u \in H^1[a,b] \mid \int_a^b u^2(x)dx = 1\}$. From the Courant-Weyl theorem we have

$$\mu_2^* = \sup_{E \in \mathcal{E}}\ \inf_{z \in E}\ Q_*[z], \quad * \in \{h,v,n\} \quad ,$$

$\mu_1^n < \mu_2^n < \ldots$ being the eigenvalues of (III.6) with Neumann boundary conditions

(BCn) $$z'(a) = z'(b) = 0 \ .$$

By observing that $c, d, u_0(a)$, $u_0(b)$, $r(a)$, $r(b)$ are positive and $g(u_0, u_0')(a)$, $g(u_0, u_0')(b) \geq 0$ it follows that

(III.7) $$\mu_i^v \leq \mu_i^n \leq \mu_i^h, \qquad i = 1, 2, \ldots \ .$$

In particular we have

(III.8) $$0 = \mu_2^v \leq \mu_2^n \leq \mu_2^h \leq \mu_3^h \ .$$

We claim that $\mu_2^h > 0$. If not, $\mu_2^n = \mu_2^h = 0$. It follows that

$$\sup_{E \in \mathcal{E}} \inf_{z \in E} Q_n[z] = \sup_{E \in \mathcal{E}} \inf_{z \in E} \{Q_n[z] + P[z]\} = 0 \ ,$$

where $P[z] := \frac{r(a)}{c} z^2(a) + \frac{r(b)}{d} z^2(b) \geq 0$.

Let us define by n the eigenfunction corresponding to μ_2^n normalized by $n(a) > 0$ and $\int_a^b n^2(x)dx = 1$.

We have $Q_n[n] \geq \inf_{z \in E} \{Q_n[z] + P[z]\}$ for every $E \in \mathcal{E}$. Let us denote by n_1 the eigenfunction corresponding to μ_1^n and define $n_1^\perp := \{u \in H^1[a,b] \mid \int_a^b u n_1 dx = 0\}$ and $E_1 := n_1^\perp \cap B$. Observe that $E_1 \subset \mathcal{E}$ and $Q_n[n] = \inf_{z \in E_1} Q_n[z]$. It is easily checked that there exists $\bar{z} \in E_1$ such that $Q_n[\bar{z}] + P[\bar{z}] = \inf_{z \in E_1} \{Q_n[z] + P[z]\}$. Consequently we have

$$Q_n[z] \geq Q_n[n] \geq Q_n[\bar{z}] + P[\bar{z}], \qquad z \in E_1 \ .$$

By taking $z = \bar{z}$ we get $P[\bar{z}] = 0$. Thus $Q_n[z] \geq Q_n[\bar{z}]$, $z \in E_1$. By observing that n is the unique function in E_1 to satisfy $Q_n[n] = \inf_{z \in E_1} Q_n[z]$, we get $n = \bar{z}$, hence $P[n] = 0$ in contradiction to $n(a) > 0$. This completes the proof of the case $c, d > 0$. Finally let $c = 0$, $d > 0$. As before we have $0 = \mu_2^v \leq \mu_2^n$. Let us call $\mu_1^d < \mu_2^d < \ldots$ the eigenvalues of (III.6) with

(BCd) $$z(a) = 0, \ z'(b) = 0 \ .$$

We prove $\mu_2^d < \mu_2^h$ exactly as before by replacing $H^1[a,b]$ by $H^1_{0,\cdot}[a,b] := \{u \in H^1[a,b] \mid u(a) = 0\}$ and $Q_h[z]$ defined by $Q_n[z] = Q_n[z] + \frac{r(b)}{d} z^2(b)$. With this new definition we have

$$\mu_2^d = \sup_{E \in \mathcal{E}} \inf_{z \in E} Q_n[z]$$

and

$$\mu_2^n = \sup_{E \in \mathcal{E}_0} \inf_{z \in E} Q_h[z] ,$$

where

$$\mathcal{E}_0 := \{E \subset H^1_{0,\cdot}[a,b] \mid E = F \cap B, \ F \text{ closed subspace of } H^1_{0,\cdot}[a,b] , \ \text{codim } F = 1\} .$$

The proof will be complete once we have shown that $\mu_2^n \leq \mu_2^d$. But this is true since

$$\mu_2^n = \sup_{E \in \mathcal{E}} \inf_{z \in E} Q_n[z] \leq \sup_{E \in \mathcal{E}_0} \inf_{z \in E} Q_n[z] = \mu_2^d .$$

IV. Discussion of condition (*). Boundary conditions and uniqueness of the asymptotic solution

The aim of this section is primarily to give sufficient conditions on g in order that there exists an unbounded continuum $\mathcal{S}$ of concave solutions of (P). We shall use the following definitions:

$$K := \{(x,u,v) \in [a,b] \times \mathbb{R}^2 \mid u - cv \geq 0, \quad u + dv \geq 0\} ,$$

$$\Omega := \{(x,u,v) \in K \mid g(x,u,v) > 0\} ,$$

$$\Omega_0 := \text{ the connected component of } \Omega \text{ containing } [a,b] \times \{(0,0)\} .$$

In the autonomous case for simplicity we shall drop the first component.

Observe that from (H1) and (H2) $\Omega_0 \neq \phi$. We shall say that $\mathcal{S}$ "$\subseteq$" Ω_0 if $(x,u(x), u'(x)) \in \Omega_0$ for each $x \in [a,b]$ and $(\lambda,u) \in \mathcal{S}$. Observe that $\mathcal{S}$ "$\subseteq$" Ω_0 implies even strict concavity. In Theorem IV.3 we give sufficient and almost necessary conditions in order that $\mathcal{S}$ "$\subseteq$" Ω_0. We show that $\mathcal{S} = \mathcal{C}$ in Corollary IV.4 for the autonomous case and in Theorem IV.5 in the nonautomonous case under stronger assumptions.

Obviously the preceding theorems give bounds on u and u' for $(\lambda,u) \in \mathcal{S}$, whenever Ω_0 is bounded.

Finally the fact that $\mathcal{S}$ "$\subseteq$" Ω_0 is exploited in Theorem IV.6 and IV.7 in order to give information on the boundary conditions satisfied by the asymptotic solutions, and the uniqueness of those solutions in Theorem IV.8. The proofs of the preceding theorems are given at the end of this section.

Theorem IV.3. *Let, in problem* (P), g *satisfy* (H1), (H2).
Let $d_{\alpha,\beta} := \{(x,u,\beta) \mid \quad u = \beta(x-\alpha), \quad x \in [a,b]\}$.
If Ω_0 *satisfies* (C1)

$$\text{(C1)} \quad \begin{cases} 1) \quad d_{\alpha,\beta} \cap \Omega_0 \text{ is connected for every } (\alpha,\beta) \in \mathbb{R}^2 , \\ 2) \quad (a,c\beta,\beta) \in d_{a-c,\beta} \cap \Omega_0 \text{ for every } \beta > 0 , \\ 3) \quad (b,-d\beta,\beta) \in d_{b+d,\beta} \cap \Omega_0 \text{ for every } \beta < 0 , \end{cases}$$

then there exists a subcontinuum $\mathcal{S} \subseteq \mathcal{C}$ *unbounded in* $\mathbb{R} \times C^1[a,b]$ *such that* $g(x,u,u') > 0$ *for each* $(\lambda,u) \in \mathcal{S}$.

Observe that in the autonomous case the condition (C1) reduces to the following one:

$$\text{(C2)}\quad \begin{cases} d_\beta \cap \Omega_0 = \{[\beta c, \delta_0[\cup \bigcup_{i=1}^{\infty}]\gamma_i, \delta_i[, \beta\} \\ \underline{\text{where}}\quad \beta c \le \delta_0 < \gamma_1 \le \delta_1 < \gamma_2 \le \delta_2 < \gamma_3 \cdots \\ \underline{\text{and}}\quad \gamma_i - \delta_{i-1} \ge |\beta|\,(b-a), \quad \underline{\text{with}}\quad d_\beta := \{(t,\beta) \mid t \in \mathbb{R}\} \end{cases}$$

In the autonomous case we have:

Corollary IV.4. <u>If in Theorem IV.3</u>, g <u>does not depend on</u> x, <u>then</u> $\mathfrak{D} = \mathfrak{C}$.

<u>Remark IV.1</u>. It should be noted that if in Theorem IV.3 the assumptions on Ω_0 do not hold, then condition (*) can be violated. Counter examples will be given before the proofs of the theorems.

<u>Theorem IV.5</u>. <u>Let in problem</u> (P) $g(x,u,v)$ <u>be of the form</u> $s(x)\, g(u,v)$ <u>with</u> $s \in C[a,b]$, $s > 0$, $g \in C^1(\mathbb{R}^2)$, $g(0,0) > 0$. <u>If</u> $s(x)\, g(u,v) = 0$ <u>and</u> $(x,u,v) \in \partial\Omega_0$ <u>imply</u> $g_u(u,v) < 0$, <u>then</u> $\mathfrak{C}$ "$\subseteq$" Ω_0 .

<u>Theorem IV.6</u>. <u>Let the assumptions of Theorem IV.3 be satisfied with</u> $c > 0$. <u>Let</u>

$$\alpha_0 := \sup\{\alpha \ge 0 \mid (a,\alpha,t\alpha) \in \Omega_0 \ \ \forall\, t \in [0, \tfrac{1}{c}]\}$$

$$\beta_0 := \underline{\text{the first positive zero of}}\ \ g(a,\beta,\tfrac{1}{c}\beta)\ .$$

<u>If</u>

1°) $\mathrm{Proj}_{\mathbb{R}}\, \mathfrak{D}$ <u>is unbounded</u>

2°) $\alpha_0 = \beta_0 < \infty$

3°) $g(a,\beta_0, t\beta_0) > 0$ for $t \in [0, \frac{1}{c}[$

<u>hold, then</u> $\displaystyle\lim_{\substack{\lambda\to\infty \\ (\lambda,u)\in\mathfrak{D}}} u(a) = \alpha_0$.

<u>Moreover if</u> $((\lambda_n, u_n))_{n\in\mathbb{N}} \subseteq \mathfrak{D}$ <u>is any sequence such that</u> $\lambda_n \to \infty$, $u_n \to \overline{u}$ <u>in the sense of Theorem II.1, then</u> $\overline{u}(a) - c\overline{u}'(a) = 0$.

<u>Remark IV.2</u>. With obvious modifications the same result holds at b .

<u>Remark IV.3</u>. If in Theorem IV.6 condition 2° or 3° is violated, the conclusion does not necessarily hold. For example a) if $g(u,v) = 1 - u^2 + v^2$ with $c = d = 1$ then

condition 2° is violated and $\bar{u}(x) = 1$ on $[a,b]$, b) if $g \in C(\mathbb{R}^2)$, $g(0,0) > 0$ and Ω_0 is $\{(u,v) \mid 0 < u < 1;\ -1 < v < 1;\ v \le 2u\} \setminus \{(u,v) \mid \frac{1}{2} \le u \le 1,\ \frac{1}{2} \le v \le 1\}$ with $a = 0$, $b = 2$, $c = \frac{1}{2}$, $d = 0$ then condition 3° is violated and

$$\bar{u}(x) = \begin{cases} \frac{1}{2}(x+1), & x \in [0,1] \\ -x + 2, & x \in]1,2] \end{cases}$$

Theorem IV.7. Let in problem (P) $g \in C^1(\mathbb{R}^2)$, $g(0,0) > 0$, $c > 0$ and $\alpha_0 := \sup\{\alpha \ge 0 \mid (\alpha, t\alpha) \in \Omega_0$ for every $t \in [0, \frac{1}{c}]\}$. Let Ω_0 satisfy the condition (C2) and $\alpha_0 < \infty$. If $\mathrm{Proj}_{\mathbb{R}} C$ is unbounded, then $\lim\limits_{\substack{\lambda \to \infty \\ (\lambda,u) \in C}} u(a) = \alpha_0$.

Remark IV.4. With obvious modifications the same result holds at b, with α_0 replaced by $\alpha_1 := \sup\{\alpha \mid (\alpha, t\alpha) \in \Omega_0,\ \forall t \in [-d^{-1}, 0]\}$.

In the autonomous case, Theorem IV.7 gives a precision of Theorem IV.6 in the case $\alpha_0 < \beta_0$.

Theorem IV.8. Let in problem (P) $g \in C^1(\mathbb{R}^2)$, $g(0,0) > 0$. Let Ω_0 satisfy the condition (C2). Assume there exists $M > 0$ such that $|v| \le M$ for $(u,v) \in \Omega_0$. Then there exist $\bar{u} \in C[a,b]$, depending only on a,b,c,d and $\partial\Omega_0$ such that $\lim\limits_{\substack{\lambda \to \infty \\ (\lambda,u) \in C}} |u - \bar{u}|_0 = 0$.

A counter example

Consider the problem

$$\text{(IV.1)} \qquad \begin{cases} -u'' = (1 - (u - \frac{1}{2})^2 - u'^2)u \\ u(0) = u(\frac{4\pi}{3}) = 0 \end{cases}$$

$f(u,v) = 1 - (u - \frac{1}{2})^2 - v^2$ is $C^1(\mathbb{R}^2)$, satisfies (H2) and $\Omega_0 = \Omega = \{(x,u,v) \in [0, \frac{4\pi}{3}] \times \mathbb{R}^2 \mid u \ge 0,\ f(u,v) > 0\}$. Here part 1) of condition (C1) is violated. It is easy to see that every solution (λ,u) of (IV.1) satisfies $|u|_0 \le \frac{3}{2}$, $|u|_1 \le 1$, hence $\mathrm{Proj}_{\mathbb{R}} C$ is unbounded. We shall show that there exists $\bar{\lambda} > 0$ such that $(\lambda,u) \in C$, $\lambda \ge \bar{\lambda}$ implies $(x,u(x),u'(x)) \notin \Omega_0$ and thus condition (*) is violated. If not, there exists a sequence $(\lambda_n,u_n) \in C$, $\lambda_n \to \infty$ $(x,u_n(x), u_n'(x)) \in \Omega_0$. The assumptions of

Theorem II.1 being satisfied, there exists $\bar{u} \in C[0,\frac{4\pi}{3}] \cap AC]0,\frac{4\pi}{3}[$, concave, satisfying

$$\text{(IV.2)} \quad \begin{cases} f(\bar{u},\bar{u}') = 0 \quad \text{a.e. on }]0,\frac{4\pi}{3}[\\ \bar{u}(0) = \bar{u}(\frac{4\pi}{3}) = 0 \ . \end{cases}$$

But the unique solution of (IV.2) belonging to $C[0,\frac{4\pi}{3}] \cap AC]0,\frac{4\pi}{3}[$ is $\bar{u}(x) = \frac{1}{2} + \sin(x - \frac{\pi}{6})$ which is not concave, contradiction.

Similarly if part 2) of condition (C1) is violated, the condition (*) may not hold. In particular if $g(u,v) = \{[20 - (u-1)(u-2)(u-3)]^2 - v^2\}\,u$, one can show that if $b - a$ is big enough condition (*) does not hold.

<u>Proof of Theorem IV.3</u>. Let g^+ be defined by

$$g^+(x,u,v) = \begin{cases} g(x,u,v) & \text{if } (x,u,v) \in \Omega_0 \\ 0 & \text{otherwise} \end{cases}$$

and consider the problem $-u'' = \lambda g^+(x,u,u')$ with the same boundary conditions as in problem (P). g^+ satisfies *(H1)* and *(H2)* and therefore *there exists a continuum of* positive solutions $\mathfrak{G}$ unbounded in $\mathbb{R} \times C^1[a,b]$. We claim that for all $(\lambda,u) \in \mathfrak{G}$ we have $(x,u(x),u'(x)) \in \Omega_0$, $x \in [a,b]$. Assume there exists $(\bar{x}, u(\bar{x}), u'(\bar{x})) \notin \Omega_0$, and let us consider the initial value problem

$$\text{(IV.3)} \quad \begin{cases} -w'' = \lambda g(x,w,w') \\ w(\bar{x}) = u(\bar{x}), \quad w'(\bar{x}) = u'(\bar{x}) \ . \end{cases}$$

Assume $\bar{\beta} := u'(\bar{x}) > 0$. Let $\bar{\alpha} := \bar{x} - \frac{u(\bar{x})}{\bar{\beta}}$. By (1) $s := d_{\bar{\alpha},\bar{\beta}} \cap \Omega_0$ is connected. If s is empty then $\bar{w}(x) = u'(\bar{x})(x-\bar{x}) + u(\bar{x})$ is the unique solution of (IV.3) on $[a,b]$ since g^+ vanishes on Ω_0^c. But $u = \bar{w}$ does not satisfy the boundary conditions, thus s is not empty, i.e. s is an interval distinct from $d_{\bar{\alpha},\bar{\beta}}$. It follows that g^+ vanishes on $d_{\bar{\alpha}\,\bar{\beta}} \backslash s$, therefore the solution of (IV.3) on $]a,\bar{x}[$ or $]\bar{x},b[$ must be $w(x) = u'(\bar{x})(x-\bar{x}) + u(\bar{x})$. In the case $\bar{\alpha} = a-c$ from 2) $u = \bar{w}$ on $[\bar{x},b]$, which is impossible since the condition $u(b) + du'(b) = 0$ is violated. If $\bar{\alpha} \neq a - c$, by a

similar argument the boundary conditions are violated at $x = a$ or $x = b$. We get the same contradiction if $\beta \leq 0$. Therefore $(\bar{x}, u(\bar{x}), u'(\bar{x})) \in \Omega_0$ for every $\bar{x} \in [a,b]$, and for each $(\lambda,u) \in \mathfrak{D}$. The proof is completed by observing that $g^+ = g$ on Ω_0, hence $\mathfrak{D}$ is a subcontinuum of $\mathcal{C}$, unbounded in $\mathbb{R} \times C^1[a,b]$.

<u>Proof of Corollary IV.4</u>. Since $\mathfrak{D}$ is a subcontinuum of $\mathcal{C}$, it is enough to show that $\mathfrak{D}$ is open in $\mathcal{C}$. Observe that for each $(\lambda,u) \in \mathfrak{D}$, there exists an open neighbourhood $\mathcal{V}(\lambda,u)$ of (λ,u) in $\mathcal{C}$ such that if $(\mu,v) \in \mathcal{V}(\lambda,u)$, then $(\mu,v) \in \Omega_0$, hence (see Theorem III.2) $(\lambda,v) \in \mathfrak{D}$. Thus $\mathfrak{D}$ is open in $\mathcal{C}$ and $\mathfrak{D} = \mathcal{C}$.

<u>Proof of Theorem IV.5</u>. Assume there exists $x \in [a,b]$ such that $(x,u(x),u'(x)) \in \Omega_0^c$. We shall show later that there exists $y \in [a,b]$ such that $(y,u'(y),u''(y)) \in \partial\Omega_0$, $u'(y) \neq 0$ and $u''(z) > 0$ for $x < z < y$ if $u'(y) > 0$, for $y < z < x$ if $u'(y) < 0$. On the other hand for such y we have $u''(z) < 0$ for $y < z < y + \delta$ if $u'(y) < 0$, respectively for $y - \delta < z < y$ if $u'(y) > 0$ and some $\delta > 0$. Indeed $u'(y) < 0$ implies

$$-u''(y+h) = \lambda s(y+h)\ [f_u(u(y),\ u'(y))\ u(y)\ u'(y)h + o(h)] > 0$$

for $0 < h < \delta$ and we get a contradiction. Similarly, if $u'(y) > 0$. Thus it remains to show the existence of such y.

First observe that if u attains its maximum at $\bar{x}$ (necessarily in $]a,b[$ from B.C) then $g(\bar{x}, u(\bar{x}),0) \geq 0$. We claim that $(\bar{x},u(\bar{x}),0) \in \bar{\Omega}_0$. Indeed from the connectedness of $\mathcal{C}$ $I := \{|u|_0 \mid (\lambda,u) \in \mathcal{C}\}$ is an interval in $\mathbb{R}^+$ containing 0. Let $u_0 \in]0,\infty]$ be the first zero of $f(u,0)$. If $u_0 = \infty$ we are done. If not by our assumptions there exists $\delta > 0$ such that $f(u,0) < 0$ if $u \in]u_0,u_0 + \delta[$. Therefore $I \subseteq [0,u_0]$, hence $(\bar{x}, u(\bar{x}),0) \in \bar{\Omega}_0$, and $\{(\tilde{x},u(\tilde{x}), u'(\tilde{x})) \mid (\lambda,u) \in \mathcal{C}, u'(\tilde{x}) = 0\} \subseteq \bar{\Omega}_0$. Let $\alpha = \min\{x \mid u(x) = |u|_0\}$, $\beta = \max\{x \mid u(x) = |u|_0\}$. We then have $u(x) = |u_0|$ for $x \in [\alpha,\beta]$. If not, u has a minimum for some $\tilde{x} \in]\alpha,\beta[$, which is impossible since $u''(x) < 0$. For the same reason, taking into account that $u'(a) > 0$, $u'(z) > 0$ for $z \in [a,\alpha[$. Similarly $u'(z) < 0$ for $z \in]\beta,b]$.

Assume there exists $x \in [a,b]$ such that $(x,u(x),u'(x)) \in \Omega_0^c$. We already know that $x \notin [\alpha,\beta]$. If $x < \alpha$, let

$$y := \min\{t \in]x,\alpha] \mid (t,u(t), u'(t)) \in \partial\Omega_0^c\} .$$

By definition, $u''(t) > 0$ for $x \leq t < y$. $y \neq \alpha$, otherwise $u(\alpha)$ would not be maximum, hence $u'(y) > 0$. The case $x > \beta$ is similar. This completes the proof of Theorem IV.5.

Proof of Theorem IV.6. First observe that $(a, \beta_0, c^{-1}\beta_0) \in \partial\Omega_0$. From Theorem IV.3, the connectedness of $\mathfrak{S}$ and the definition of β_0, if $(\lambda,u) \in \mathfrak{S}$, then: $u'(a) < \frac{1}{c}\beta_0$ and $u(a) \leq \beta_0$. From the boundary conditions at a, and the concavity of u, it follows that $\sup_{(\lambda,u)\in\mathfrak{S}} |u|_0 < \infty$. In order to complete the proof, it is sufficient to show that if $(\lambda_n,u_n) \in \mathfrak{S}$, $\lambda_n \to \infty$, $u_n \to \bar{u}$, $u_n' \to \bar{v}$ as in Theorem II.1, then $\bar{u}(a) = \alpha_0$, and $\bar{u}(a) = c\bar{u}'(a)$. From the definition of α_0 and assumptions 2 and 3 we have: $g(a,u,v) = 0$, $(u,v) \in \Delta$ imply $u = \alpha_0$ and $u = cv$, where $\Delta := \{(u,v) \mid u - cv \geq 0,\ u \leq \alpha_0,\ v \geq 0\}$. From the proof of Theorem II.1 we have $g(a,\bar{u}(a), \bar{u}'(a)) = 0$, $0 \leq \bar{u}'(a) \leq \bar{v}(a)$ and $\bar{u}(a) = c\bar{v}(a)$. As $\bar{u}(a) \leq \alpha_0$ the proof is complete.

Proof of Theorem IV.7. As in the proof of Theorem IV.6 it is sufficient to show that $u(a) \leq \alpha_0$ when $(\lambda,u) \in \mathfrak{C}$. Assume there exists $(\tilde{\lambda},\tilde{u}) \in \mathfrak{C}$ such that $\tilde{u}(a) > \alpha_0$. By connectedness of $\mathfrak{C}$, $\{u(a) \mid (\lambda,u) \in \mathfrak{C}\} \supseteq [0,\tilde{u}(a)] \supseteq [0,\alpha_0]$. Thus there exists $(\hat{\lambda},\hat{u}) \in \mathfrak{C}$ such that $\hat{u}(a) = \alpha_0$. From Theorem III.2, there exists a C^1 function $s \to (\lambda(s), u(s)) \in \mathbb{R} \times C^2[a,b]$, $s \in [0,1]$ such that $(\lambda(0), u(0)) = (\lambda_1,0)$, $(\lambda(1), u(1)) = (\hat{\lambda},\hat{u})$. From the concavity of $\hat{u}$ we have

$$\alpha_0 \leq \hat{u}(a) \leq \hat{u}(x) \leq |\hat{u}|_0 \quad \text{as long as} \quad \hat{u}'(x) \geq 0 .$$

By definition of α_0 and the fact that $\alpha_0 < \infty$ there exists $0 \leq \tilde{v} \leq c^{-1}\alpha_0$ such that $(\alpha_0,\tilde{v}) \in \partial\Omega_0$. By a homotopy argument there exists $\bar{s} \in]0,1]$ and $\bar{x} \in [a,b]$ such that $(u(\bar{s})(\bar{x}),\ u'(\bar{s})(\bar{x})) \in \partial\Omega_0$ which contradicts the fact that if $(\lambda,u) \in \mathfrak{C}$ then $(u(x), u'(x)) \in \Omega_0$, $x \in [a,b]$.

Proof of Theorem IV.8. From Corollary IV.4, $\mathfrak{C}$ "$\subseteq$" Ω_0. By assumption $\exists\, M' > 0$ such that $|u|_0, |u|_1 \leq M'$ for $(\lambda,u) \in \mathfrak{C}$. Thus $\mathrm{Proj}_{\mathbb{R}}\mathfrak{C}$ is unbounded. Let $(\lambda_n,u_n)_{n\in\mathbb{N}} \subseteq \mathfrak{C}$ and $\bar{u} \in C[a,b]$, $\bar{u} \neq 0$, such that $\lambda_n \to \infty$ and $\lim_{n\to\infty} |u_n-\bar{u}|_0 = 0$. Such a $\bar{u}$ exists by Theorem II.1. If $c = 0$, $\bar{u}(a) = 0$ and if $d = 0$, $\bar{u}(b) = 0$. (See Remark II.1.b and c). If $c > 0$, observe that $\alpha_0 = \sup\{\alpha \mid (\alpha,t\alpha) \in \Omega_0,\ \forall\, t \in [0,\frac{1}{c}]\}$

satisfies $\alpha_0 \le cM$ and depends only on c and $\partial\Omega_0$. By Theorem IV.7 $\bar{u}(a) = \alpha_0$. Similarly, if $d > 0$ $\bar{u}(b) = \alpha_1$ (α_1 of Remark IV.4) with $\alpha_1 \le dM$ and depending only on d and $\partial\Omega_0$. Let u_0 be the first positive zero of $g(u,0)$.

Let $\mathfrak{J} := \{h \in C[0,u_0] \mid h \ge 0,\ h$ decreasing, $h(u_0) = 0,\ h(u) = h(0)$ for $0 \le u \le ch(0)$; $(u,h(u) \in \Omega_0$ for $u \in [ch(0), u_0[$; $\{(u,v) \mid u - cv \ge 0,\ 0 \le v \le h(u)\} \subseteq \Omega_0\}$.

Note that $h = 0$ belongs to $\mathfrak{J}$. Moreover if $(\lambda,u) \in \mathcal{C}$, $u \neq 0$, then the function h defined by

$$h(u) := \begin{cases} u'(0) & \text{for } 0 \le u \le u(0) \\ u'(x) & \text{for } u(0) \le u(x) \le |u|_0 \\ 0 & \text{for } |u|_0 \le u \le u_0 \end{cases}$$

belongs to $\mathfrak{J}$ by using the fact, that $\mathcal{C}$ is a curve. Let $\bar{\beta}(u) \le \sup\{h(u) \mid h \in \mathfrak{J}\}$, $u \in [0,u_0]$. $\bar{\beta}$ is bounded by M, decreasing and l.s.c. hence $\bar{\beta}(u^+) = \bar{\beta}(u)$. Let $\beta(u) := \{(u,[\bar{\beta}(u^+), \bar{\beta}(u^-)]) \mid u \in [0,u_0]\}$ be the maximal decreasing graph associated with $\bar{\beta}$.

We claim that if $(\hat{u},\hat{v}) \in \partial\Omega_0$ and $\hat{v} \ge 0$ then $\bar{\beta}(\hat{u}^+) \le \hat{v}$. If not $\hat{v} < \bar{\beta}(\hat{u}^+) = \bar{\beta}(\hat{u})$. From the definition of $\bar{\beta}$ there exists $h \in \mathfrak{J}$ such that $\hat{v} < h(\hat{u}) < \bar{\beta}(\hat{u})$. But

$$(\hat{u},\hat{v}) \in \{(u,v) \mid u-cv \ge 0,\quad 0 \le v \le h(a)\} \subseteq \Omega_0 ,$$

a contradiction. Let $[\check{x},\hat{x}]$ be the maximal interval in $[a,b]$ such that $\bar{u}(x) = |\bar{u}|_0$. We shall prove that $\dot{\bar{u}} \in \beta(\bar{u})$ a.e. on $[a,\hat{x}]$. If $\check{x} < \hat{x}$, then $\bar{u}'(x) = 0$ on $]\check{x},\hat{x}[$, hence $g(\bar{u}(x),0) = 0,\ x \in]\check{x},\hat{x}[$. It follows that $\bar{u}(x) = u_0$ and thus $\dot{\bar{u}} \in \beta(\bar{u})$ on $]\check{x},\hat{x}[$. If $a = \check{x}$ we are done, otherwise let $A := \{x \in]a,\check{x}[\mid \bar{u}'_+(x) = \bar{u}'_-(x)\}$. Observe that $]a,\check{x}[- A$ is a countable set and $\bar{u}'(x) > 0,\ x \in A$. Let $x \in A$. Since $\bar{u}'(x)$ exists and is positive, for n big enough $u_n'(x) > 0$.

Since $(\bar{u}(x), \bar{u}'_+(x)) \in \partial\Omega_0$ we have $\bar{\beta}(\bar{u}(x)) \le \bar{u}'_+(x) = \bar{u}'(x)$. For n big enough we have $u_n'(x) \le \bar{\beta}(u_n(x))$, hence $\bar{u}'(x) = \bar{v}(x) \le \bar{\beta}(\bar{u}(x)^-)$. Thus $\bar{u}'(x) \in \beta(\bar{u}(x))$. Consequently $\bar{u}$ is a solution of

$$\begin{cases} u + (-\beta(u)) \ni 0, & \text{a.e. } x \in [a,\hat{x}] , \\ u(a) = \bar{u}(a) . \end{cases}$$

By observing that $-\beta$ is a maximal monotone graph in the sense of [2], this equation

has exactly one solution, hence $\overline{u}$ is uniquely defined by a,c. and $\partial\Omega_0$ on $[a,\hat{x}]$. Note that from [2] we have $\overline{u}'_+(x) = \overline{\beta}(\overline{u}(x))$ for every $x \in [a,\hat{x}[$.

By proceeding in the same fashion on $[\check{x},b]$, $\overline{u}$ is uniquely defined on $[\check{x},b]$ by b,d and $\partial\Omega_0$. Since $\check{x} \leq \hat{x}$ we are done.

V. Examples

1°. Let $f \in C^1(\mathbb{R})$, $f(\rho) > 0$ for $\rho \in [0,\rho_0[$, $f(\rho_0^2) = 0$, and consider the following problem:

$$(P1) \qquad \begin{cases} -u'' = \lambda f(u^2 + u'^2)u & \text{on }]0,T[\\ u(0) = u(T) = 0 . \end{cases}$$

From Corollary IV.4, it follows that the condition (*) is satisfied which implies by Theorem III.2 that $\mathcal{C}$ is a curve in $\mathbb{R} \times C^2[0,T]$. Moreover, u is bounded in $C^1[0,T]$ for $(\lambda,u) \in \mathcal{C}$. From Theorem IV.8, $\lim\limits_{\substack{\lambda\to\infty \\ (\lambda,u)\in\mathcal{C}}} |u-\bar{u}_T|_0 = 0$ where

$$\bar{u}_T(x) := \begin{cases} \rho_0 \sin x & x \in [0, \min(\frac{\pi}{2}, \frac{T}{2})] \\ \rho_0 & x \in]\min(\frac{\pi}{2},\frac{T}{2}), \max(\frac{T}{2}, T-\frac{\pi}{2})[\\ \rho_0 \sin(T-x) & x \in [\max(\frac{T}{2}, T-\frac{\pi}{2}), T] \end{cases}$$

Remarks.

1. If instead of $f(u^2 + u'^2)$ we have $s(x)\, f(u^2 + u'^2)$, $s \in C[0,T]$, $s > 0$, all the previous conclusions hold except the fact that $\mathcal{C}$ is a curve, which we can not decide.
2. If $T = \pi$, obviously $\mathcal{C} := \{(\frac{\alpha}{f(\alpha^2)}, \alpha \sin x) \mid 0 < \alpha < \rho_0\}$ and $\bar{u}_\pi(x) = \rho_0 \sin x$ is smooth but if $T < \pi$, $\bar{u}_T \notin C^1[0,T]$.
3. Observe that the limit $\bar{u}_T$ in Remark 1, is independent of s and depends only on the first zero of f and on T.
4. If f does not vanish and $\mathrm{Proj}_{\mathbb{R}}\, \mathcal{C}$ is unbounded, then $|u|_0$ is not bounded for $(\lambda,u) \in \mathcal{C}$. This is an immediate consequence of Theorem II.1.

2°. Let f be as in example 1°, and consider

$$(P2) \qquad \begin{cases} -u'' = \lambda f((u - \frac{\rho_0}{2})^2 + u'^2)u & \text{on }]0,T[\\ u(0) - cu'(0) = 0 & c,d \geq 0 \\ u(T) + du'(T) = 0 . \end{cases}$$

The case $f(\rho) = 1-\rho$, $c = d = 0$, $T = \frac{4\pi}{3}$ has been treated in Section IV. If $c \in [0, \frac{1}{2}[$

or $d \in [0, \frac{1}{2}[$, by using a similar argument as in Section IV, one can prove that the condition (*) is violated. However, if $c,d \in [\frac{1}{2}, \infty[$, the condition (*) holds by Corollary IV.4 and $\mathcal{C}$ is a curve in $\mathbb{R} \times C^2[0,T]$. From Theorems IV.8 and IV.6, $\lim_{\substack{\lambda\to\infty \\ (\lambda,u)\in\mathcal{C}}} |u-\bar{u}|_0 = 0$ where $\bar{u} \in C[0,T]$, satisfies the boundary conditions and $f(\bar{u}(x), \bar{u}'_+(x)) = 0$ $x \in [a,b]$.

3°. Let $f(x,u,u') = f(x,u)$, $f \in C([a,b] \times \mathbb{R})$, $f(x,0) > 0$, $x \in [a,b]$ and consider:

$$
(P3) \qquad \begin{cases} -u'' = \lambda f(x,u)u, & x \in]a,b[, \\ u(a) - cu'(a) = 0, & c,d \geq 0 , \\ u(b) + du'(b) = 0 . & \end{cases}
$$

It follows from Theorem II.1, that a necessary condition in order that $|u|_0 \leq M$, u concave for every $(\lambda,u) \in \mathcal{C}$ is that there exists $\bar{u} \in C\,]a,b[$, concave such that $\{(x,\bar{u}(x), \bar{u}'_+(x)) \mid x \in]a,b[\} = \partial\Omega_0$. Therefore we shall assume that this condition holds, i.e., if for $x \in]a,b]$, $\bar{u}(x)$ is the first positive zero of $u \to f(x,u)$, then $\bar{u}(x)$ is concave. By Theorems IV.3 and II.1, we get the existence of a subcontinuum $\mathcal{D} \subseteq \mathcal{C}$ such that $\lim_{\substack{\lambda\to\infty \\ (\lambda,u)\in\mathcal{C}}} |u-\bar{u}|_{0,[a',b']} = 0$ for every $a < a' < b' < b$. We do not know in general if this holds for the whole continuum $\mathcal{C}$. However if $f(x,u) \leq 0$ for $u \geq \bar{u}(x)$, $f(x,\cdot) \in C^1(\mathbb{R})$ and $\bar{u}(a) - c\bar{u}'(a^+) \geq 0$, $\bar{u}(b) + d\bar{u}'(b^-) \geq 0$ hold, then $\mathcal{D} = \mathcal{C}$. In this case observe that if $(\lambda,u) \in \mathcal{C}$ and $x \in [a,b]$ is such that $(x,u(x),u'(x)) \in \partial\Omega_0$ then there exists $\bar{x} \in [a,b]$ such that $(\bar{x}, u(\bar{x}), u'(\bar{x})) \in \bar{\Omega}_0^c$. It follows that $\mathcal{C}$ is the disjoint union of $\{(\lambda,u) \in \mathcal{C} \mid (x,u(x),u'(x)) \in \Omega_0\ \forall\ x \in [a,b]\}$ and $\{(\lambda,u) \in \mathcal{C} \mid \exists\ x \in [a,b]$ such that $(x,u(x),u'(x)) \in \bar{\Omega}_0^c\}$ which are open in $\mathcal{C}$.

By the connectedness of $\mathcal{C}$, we are done.

In the case $f(x,u) = s(x)\,h(u)$, $s \in C[a,b]$, $s > 0$, $h \in C^1[a,b]$, $\mathcal{D} = \mathcal{C}$ also. Indeed $\{|u|_0 \mid (\lambda,u) \in \mathcal{C}\}$ is an interval containing 0 by the connectedness of . If there exists $(\lambda,u) \in \mathcal{C}$ such that $f(|u|_0) = 0$ then $u(x) = |u|_0$ is the unique solution of the corresponding initial value problem at $\bar{x} \in [a,b]$ where $u(\bar{x}) = |u|_0$. But $u = |u|_0$ does not satisfy the boundary conditions. Observe that in this case the

condition $f(x,u) \le 0$ for $u \ge \bar{u}(x)$ is not necessary. Finally if $s(x) \equiv 1$, $\mathcal{C}$ is a curve in $\mathbb{R} \times C^2[a,b]$.

4°. Let $f(x,u,v) = f(x,v)$, $f \in C([a,b] \times \mathbb{R})$, $f(x,0) > 0$ and consider

$$(P4) \qquad \begin{cases} -u'' = \lambda f(x,u')u \ , \\ u(0) = u(1) = 0 \ . \end{cases}$$

Assume that there exists $\bar{v}_+(x)$ positive decreasing and $\bar{v}_-(x)$ negative decreasing such that $f(x,v) > 0$ for $\bar{v}_- < v < \bar{v}_+$ and $f(x,\bar{v}_\pm(x)) = 0$. Observe that the condition (C1) of Theorem IV.3 is satisfied. It follows in particular that $|u|_1$, hence $|u|_0$ is bounded for $(\lambda,u) \in \mathcal{D}$ where $\mathcal{D}$ is a subcontinuum of $\mathcal{C}$ such that $\mathrm{Proj}_{\mathbb{R}}\mathcal{D}$ is unbounded. Theorem II.1 can be applied to $\mathcal{D}$. In the case $f(x,v) = s(x)\ h(v)$, $s \in c[a,b]$, $s > 0$, $h \in C^1$ where v_- and v_+ are the first negative, resp. positive zeros of h , $\mathcal{D} = \mathcal{C}$. As in 3°, note that v_- (resp. v_+) is a solution of the differential equation (P4) without satisfying the boundary conditions. Moreover if $s(x) = 1$, $\mathcal{C}$ is a curve in $\mathbb{R} \times C^2[a,b]$. An interesting subcase is treated in 5°.

5°. Consider the problem

$$(P5) \qquad \begin{cases} -u'' = \lambda f(u')u, \quad \text{on }]a,b[, \\ u(a) = u(b) = 0 \ . \end{cases}$$

Theorem V.9. Let f in problem (P5) satisfy $f \in C^2(\mathbb{R})$, $f(0) > 0$, $f''(x) \le 0$ and f is not constant on any neighborhood of 0 . (For example, if f' has at most one zero). Then a) Every positive solution of (P5) belongs to $\mathcal{C}$. b) $\mathcal{C}$ is a curve parametrized by $\lambda \in [\lambda_1,\infty[$, $[\lambda_1,\infty)$, c) $\lim_{\substack{\lambda\to\infty \\ (\lambda,u)\in\mathcal{C}}} |u-\bar{u}|_{0,[a',b']} = 0$ for every $a < a' < b' < b$. where $\bar{u}$ is the unique concave positive solution of $f(u') = 0$ $x \in]a,b[$, satisfying $\bar{u}(a) = 0$ (resp. $\bar{u}(b) = 0$) if f has a positive zero (resp. negative zero). Moreover a' can be chosen equal to a if $\bar{u}(a) = 0$ (resp. $b' = b$, if $\bar{u}(b) = 0$) .

Remark. It is easy to see that every solution of (P5) can be deduced from a positive one (resp. a negative one) by observing that the zeros of any solution of (P5) are equidistant, and thus correspond by an appropriate "stretching" to a positive or a negative

one. The negative solutions are obtained from the positive ones by antisymmetry. It follows in particular that if $\lambda \leq \lambda_1$, (P5) has only the trivial solution 0, and if $\lambda_k < \lambda \leq \lambda_{k+1}$ where λ_k are the eigenvalues of the linearized problem at the origin, then (P5) has exactly $2k-1$ solutions. It follows that every solution of (P5) are contained in the continua $C_k^{\pm}$ defined in [10], which here are curves in $\mathbb{R} \times C^2[a,b]$ and moreover parameterized by λ. The asymptotic behavior as $\lambda \to \infty$ for the $C_k^{\pm}$ is thus easily deduced from that of C_1^{+}.

Proof of Theorem V.9. From example 4° we know that $\mathcal{C}$ is a curve in $\mathbb{R} \times C^2[a,b]$. In order to prove that $\mathrm{Proj}_{\mathbb{R}} \mathcal{C}$ is unbounded, we shall need the following

Lemma V.1. Let $g(x,u,v)$ satisfying (H1) and (H2). Assume there exists an increasing function $c: \mathbb{R}^+ \to \mathbb{R}^+$ such that $g(x,u,v) \leq c(u)\ (1 + |v|)$ for $u > 0$ and $x \in [a,b]$.

If in problem (P) corresponding to g, the condition (*) holds and if $(x,u,v) \in \Omega_0$ implies $v \leq M$ (resp. $v \geq -M$), then $|u|_0$ is bounded for $(\lambda,u) \in \mathcal{C}$ and $\mathrm{Proj}_{\mathbb{R}} \mathcal{C}$ is unbounded.

Proof of Lemma V.1. The fact that $|u|_0$ is bounded for $(\lambda,u) \in \mathcal{C}$ follows immediately from the boundary conditions and the half bound on u'. Assume $\mathrm{Proj}_{\mathbb{R}} \mathcal{C}$ is bounded. Since $-u''(x) \geq 0$ and $|u|_0$ is bounded for $(\lambda,u) \in \mathcal{C}$ $\int_a^b |u'(x)|dx$ is bounded. Using the assumption on g we get $\int_a^b |u''(x)|dx \leq M' \int_a^b (1 + |u'|)dx$ is bounded, hence by $-u'' \geq 0$, $|u|_1$ is bounded which contradicts the fact that $\mathcal{C}$ is unbounded in $\mathbb{R} \times C^1[a,b]$.

Let us return to the proof of Theorem V.9. Let us denote by v^- (resp. v^+) the first negative (resp. positive) zero of f, possibly infinite. Observe that our assumptions on f imply that v^- or v^+ is finite. Moreover, from example 4°, we know that condition (*) is satisfied. Since f is concave, the assumptions of Lemma V.1 are satisfied, then $|u|_0$ is bounded for $(\lambda,u) \in \mathcal{C}$ and $\mathrm{Proj}_{\mathbb{R}} \mathcal{C}$ is unbounded. Note that Theorem II.1 gives c). In order to show that is parametrized by λ, it is enough to prove that if $(\lambda_0,u_0) \in \mathcal{C}$, $(\lambda(s), u(s))$, $s \in]-1,+1]$ is as in Theorem III.2 and $\lambda_s(0) = 0$, then $\lambda_{ss}(0) > 0$. [Remark that $\mathcal{C}$ is a C^2 one dimensional manifold in $\mathbb{R} \times C^2[a,b]$ since $f \in C^2(\mathbb{R})$]. Indeed for $s = 0$, we have:

$$-u'' - \lambda f(u')u = 0, \quad u(a) = u(b) = 0 ,$$

$(P_s 5)$ $$-(ru_s')' - \lambda r\, f(u')u_s = 0, \quad u_s(a) = u_s(b) = 0 \quad ,$$

$(P_{ss}5)$ $$\begin{cases} -(ru_{ss}')' - \lambda r\, f(u')\, u_{ss} = \lambda_{ss}\, r\, f(u')u + \lambda r\, f''(u')\, uu_s'^2 \\ + 2\lambda r\, f'(u')\, u_s u_s', \quad u_{ss}(a) = u_{ss}(b) = 0 , \end{cases}$$

where $r(x) := \exp\{\lambda \int_a^x f'(u')(t)\, u(t)\, dt\}$ and $_s$ denotes the differentiation with respect to s. From the proof of Theorem III.2, we know that $u_s > 0$ or $u_s < 0$ on $]a,b[$. It is standard that the right hand side of $(P_{ss}5)$ has to satisfy $0 = \lambda_{ss} \int_a^b r\, f(u')\, u\, u_s\, dx + \lambda \int_a^b r\, f''(u')\, u\, u_s\, u_s'^2\, dx + 2\lambda \int_a^b r\, f'(u')\, u_s^2\, u_s'\, dx$. Hence from the assumptions on f and an integration by part on the last term, we get $\lambda_{ss} > 0$. This proves b). It remains to prove a). Assume $v^+ < \infty$ the case $v^- < \infty$ being similar. Let $(\hat\lambda, \hat u)$ be a positive solution of (P5). $\hat u'(a) < v^+$, otherwise if $\hat u'(a) \geq v^+$ there exists $\bar x \in [a,b]$ such that $\hat u'(\bar x) = v^+$, since $\hat u'$ vanishes where $\hat u$ achieves its maximum. But then $\tilde u(x)$ such that $\tilde u(\bar x) = \hat u(\bar x)$, $\tilde u'(x) = \hat u'(\bar x)$, $x \in [a,b]$ would be a solution of the initial value problem at $\bar x$, which does not satisfy the boundary conditions and which must be equal to u from the uniqueness of the solution of the initial value problem. Thus $\hat u'(a) < v^+$. On the other hand, $\{u(a) \mid (\lambda,u) \in \mathcal{C}\}$ is an interval from the connectedness of $\mathcal{C}$, which contains 0 and such that v^+ is a limit point. Indeed $f(\bar u_+'(a)) = 0$, where $\bar u$ is defined in c), thus $\bar u_+'(a) = v^+$. Moreover it follows from Theorem II.1 that in this case $\lim_{\substack{\lambda\to\infty \\ (\lambda,u)\in\mathcal{C}}} u'(a) = \bar u_+'(a) = v^+$. Consequently $\{u(a) \mid (\lambda,u) \in \mathcal{C}\} = [0,v^+[$. Finally observe that (λ,u) is a solution of (P5) iff $z_\lambda(x) := \sqrt{\lambda}\, u(\frac{x}{\sqrt{\lambda}})$ is a solution of (P'5)

(P'5) $$\begin{cases} -w'' = f(w')w , \\ w(0) = w(\sqrt{\lambda}) = 0 . \end{cases}$$

Since $z_\lambda'(0) = u'(0) \in [0,v^+[$, by the uniqueness of solutions of (P'5), $(\lambda,u) \in \mathcal{C}$ and we are done.

Remark. The condition $f'' \leq 0$ is only needed on the interval $[v^-,v^+]$.

6. Let $f(x,u,u') = s(x)\, h_1(u')\, h_2(u)$ such that $s \in C[a,b]$, $s > 0$, $h_1, h_2 \in C^1(\mathbb{R})$, $h_1(0)\, h_2(0) > 0$, and consider

$$
\text{(P6)} \qquad \begin{cases} -u'' = \lambda s(x)\, h_1(u')\, h_2(u) u \;, \\ u(a) - cu'(a) = 0, \quad c,d \geq 0 \;, \\ u(b) + du'(b) = 0 \;. \end{cases}
$$

First observe that the condition (*) is satisfied. This is proved as in examples 3° and 4°. (If $s(x) \equiv 1$, $\mathcal{C}$ is a curve). Let v^+ (resp. v^-) be the first positive (resp. negative) zero of h_1, possibly infinite. If v^+ and v^- are both finite, we can apply Theorem II.1. If $v^+ < \infty$, $v^- = -\infty$ and $h_1(v) \leq c(1 + |v|)$ for $v > 0$, by by Lemma V.1, we can apply Theorem II.1 again. Similarly if $v^- > -\infty$, $v^+ = +\infty$.

7. As a last example we consider the problem

$$
\text{(P7)} \qquad \begin{cases} -u'' + |u|^{p-1}\, u = \lambda u, \quad p > 1 \;, \\ u(0) - cu'(0) = 0, \quad c,d \geq 0 \;, \\ u(1) + du'(1) = 0 \;. \end{cases}
$$

Define $\tilde{u} = \lambda^{\frac{1}{1-p}} u$ and (P7) is equivalent to

$$
\text{(P7)}' \qquad \begin{cases} -\tilde{u}'' = \lambda\{1 - |\tilde{u}|^{p-1}\}\, \tilde{u} \;, \\ \tilde{u}(0) - c\, \tilde{u}'(0) = 0, \quad c,d \geq 0 \;, \\ \tilde{u}(1) + d\, \tilde{u}'(1) = 0 \;. \end{cases}
$$

From example 3° we have that $\mathcal{C}$ of problem (P7) is a curve in $\mathbb{R} \times C^2[0,1]$, parameterized by λ [1], such that $\lim\limits_{\substack{\lambda \to \infty \\ (\lambda,u) \in \mathcal{C}}} \lambda^{\frac{1}{1-p}} u = 1$ uniformly on every compact subinterval of $]0,1[$.

Remark. Examples 3,4,6 can be used for the forced case by considering:

$$
-u'' = \lambda g(x,u,u') \quad \text{with the boundary conditions.}
$$

Appendix.

We recall some basic results on concave functions. If u is concave on $[a,b]$, then u'_+ is l.s.c. and decreasing, u'_- is u.s.c. and decreasing. Moreover

(A1) $$u'_+(x) = \lim_{y>x} u'_+(y) \quad (\text{possibly } \infty), \quad x \in [a,b[$$

(A2) $$u'_-(x) = \lim_{y<x} u'_-(y) \quad (\text{possibly} -\infty), \quad x \in]a,b] .$$

Proposition A: Let $(u_n)_{n \in \mathbb{N}} \subseteq C^2[a,b]$, $|u_n|_0, |u_n|_1 \leq M$ u_n concave for all n. Then exist a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ and $\bar{u}, \bar{v}$ such that:

(a) $$u_{n_k} \to \bar{u} \text{ in } AC[a,b] ,$$

(b) $$u_{n_k}(x) \to \bar{v}(x), \quad x \in [a,b] ,$$

(c) $$\bar{u}'_+(x) = \bar{u}'_+(x^+) = \bar{v}(x^+) \leq \bar{v}(x) \leq \bar{v}(x^-) = \bar{u}'_-(x^-)$$
$$= \bar{u}'_-(x), \quad x \in [a,b] .$$

Proof. (a) Since $u_n \in C^2[a,b]$ and is concave, it follows that $|u''_n|_{L^1} \leq 2|u_n|_1$, hence u_n is bounded in $W^{2,1}[a,b]$. We can extract a subsequence, still denoted by u_n, converging to $\bar{u}$ in $AC[a,b]$.

(b) Since u_n is bounded in $W^{2,1}[a,b]$, we can apply Helly's Theorem [9] to $(u'_n)_{n \in \mathbb{N}}$.

(c) $\bar{u}'_+(x) = \bar{u}'(x^+)$ is (A1), since $\bar{u}$ is concave. $\bar{v}(x^+) \leq \bar{v}(x)$ since $\bar{v}$ is decreasing. From (a) and (b), $\bar{u}'_+(y) = \bar{v}(y)$ a.e. in $[a,b]$. Let $x \in [a,b]$ and $x_n \downarrow x$ such that $\bar{u}'_+(x_n) = \bar{v}(x_n)$. $\bar{v}(x^+) = \lim_{x_n \downarrow x} \bar{v}(x_n) = \lim_{x_n \downarrow x} \bar{u}'(x_n) = \bar{u}'_+(x^+)$.

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lambda   is an element of   lambda
following

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~~We study~~ the eigenvalue problem is studied:

$$
(P)\quad \begin{cases} -u'' = \lambda g(x,u,u'),\ x \in\ ]a,b[,\ u \in C^2[a,b],\ \lambda > 0\ ; \\ u(a) - cu'(a) = 0 \text{ with } c,d \geq 0\ ; \\ u(b) + du'(b) = 0\ . \end{cases}
$$

is an element of C(superscript 2)

~~We investigate~~ the continuation and the asymptotic behaviour as $\lambda \to \infty$ ~~of~~ for the positive and concave solutions ~~of (P)~~ are investigated.

lambda approaches infinity

> or =

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